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1974 J. Phys. A: Math. Nucl. Gen. 7 465

(http://iopscience.iop.org/0301-0015/7/4/010)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 4, 1974. Printed in Great Britain. © 1974

Constitutive equations for heat conduction in general relativity

Gérard A Maugin

Université de Paris (VI), Département de Mécanique Théorique (ERA du CNRS)
Tour 66, 4 Place Jussieu, 75230 Paris Cedex 05, France

Received 25 May 1973, in final form 3 October 1973

Abstract. A heat flux constitutive equation is derived in three approximations from a general *functional* constitutive equation which describes heat conduction in so-called 'simple' thermodeformable media in general relativity. The three approximations correspond to materials having a so-called 'fading memory', an 'infinitely short memory', and materials of the 'rate-type', respectively. The first approximation leads to an integral constitutive equation which, after inversion of the integral operator, yields a differential law that: (i) exhibits the *relaxation* process needed to guarantee a propagation of heat disturbances at a speed smaller than that of light; (ii) is essentially spatial; (iii) satisfies the requirements now imposed in continuum physics, in particular, the *principle of objectivity* as formulated by the author or the *rheological invariance* of Oldroyd. The equation obtained has the same three-dimensional limit as the spatial part of Kranys' equation for rigid heat conductors. However, Kranys' equation was not objective. The second approximation leads to a heat *retardation* process prohibited by the second principle of thermodynamics. The third approximation may contain the other two as particular cases. Within the frame of the approximations made for *isotropic* materials, it is shown that interactions between the different transport phenomena, eg, heat flow and viscosity, cannot be accounted for.

1. Introduction

A current paradox of classical physics is that, if one assumes Fourier's law of conduction, then heat propagates instantaneously, thus at infinite speed. The situation is even worse in relativistic physics, for perturbations of any physical field should not propagate at a velocity greater than that of light. Indeed, if a direct covariant generalization of Fourier's law as first indicated by Eckart (1940) (and also proposed by Bressan 1967) is supposed, then it can be shown, by studying the Cauchy problem for such a general relativistic heat conducting fluid (cf Marle 1969), that the characteristic manifolds are space-like hypersurfaces. This implies a signal propagation at a speed greater than c, the light velocity in vacuum, a fact in direct contradiction with the basic hypotheses of relativity physics. To remedy this undesirable fact, Kranys (1966a, b, 1967) has proposed a relativistic generalization of the heat conduction law proposed earlier by Cattaneo (1948, 1958) and Vernotte (1958) with a view to solving the same paradox but within the framework of classical physics. The modified law proposed by Kranys who introduces a relaxation of heat flux proves to be satisfactory from a purely mathematical viewpoint. Mahjoub (1971a, b) (also Boillat 1971) has shown that the characteristic manifolds of the corresponding general relativistic Cauchy problem were time-like, hence eliminating any risk of propagation at infinite velocity. In fact, it is shown that the system of equations obtained when one takes account of Kranys' law either is not strictly hyperbolic in the sense of Leray and then admits a solution in a certain Gevrey class of functions, or is strictly hyperbolic and then admits a unique solution in a Sobolev class of functions. Further work on heat conduction with Kranys' law as its starting point was done by Kranys himself (1972a, b). Also, in an excellent monograph, Stewart (1971) demonstrated by a kinetic-theory argument that the maximum phase velocity associated with heat conduction phenomena basically described by an equation of the Kranys' type was of the order of $0{\cdot}8c$.

However we shall not use any of the arguments of kinetic theory in this paper as its purpose is not to find such an upper bound for the propagation of heat disturbances. The subsequent development is drawn along a purely *phenomenological* line. It is of course preferable that both attitudes, the kinetic theory and the phenomenological approaches, yield results which are in agreement. Yet we have the firm belief that theoretical *continuum* physics is now on solid ground, almost axiomatized as much as geometry (cf Truesdell and Noll 1965), with a whole body of fundamental concepts, axioms and theorems deduced from the latter, thus forming a true mathematical physics. It needs only to be supplemented with experiments in order to measure the phenomenological constants introduced in the formulation. The same is true of general relativistic continuum physics in which we have tried to introduce a general formalism and to construct the corresponding adequate principles (Maugin 1972a, b, c, 1973a, b, c, e, f). For instance, constitutive equations—of which the heat conduction law constitutes an example—which are needed to close the underdetermined system of field equations, are supposed to satisfy several *principles* of formulation (in classical continuum mechanics, see Eringen 1967 chap 5; in relativistic physics, Maugin 1973f). Among these principles (or requirements), that of *material frame indifference*, or *objectivity*, or *rheological invariance* according to the different authors, is certainly that which bears the most profound physical significance. It expresses the fact that any constitutive equation which illustrates the response of a material to a solicitation—in the case of heat conduction, the response to a deviation from thermodynamical equilibrium—should be the same for all observers. This is not only a requirement of form invariance (covariance) that would be automatically satisfied by use of tensorial analysis. It also imposes a certain function—or *functional*—dependence on the constitutive dependent variables (eg, the heat flux vector) and dictates what independent arguments can or cannot be used in the constitutive equations. Making use of the notion of an observer, the statement of material frame indifference in general relativity can only take a local form (along the worldline of such an observer). Two such statements have been proposed independently, one by Oldroyd (1970), the other by the author (Maugin 1972a, 1973f, g). Kranys' equation satisfies none. Moreover, from our viewpoint, Kranys' proposal is not entirely satisfactory for two other reasons: (i) instead of postulating a law as Kranys did, it should be possible to deduce a satisfactory heat conduction law from a general theory of constitutive equations in relativistic continuum physics; (ii) Kranys (1966a) (also Boillat 1971) considers a four-dimensional conduction law which is not space-like. It is our opinion that he so introduces too many unknowns, ie, the fourth component of his law. He justifies his viewpoint by saying that there could exist a nonzero invariant density of heat but we do not follow his argument.

The aim of the present study is therefore clear: to deduce from a general functional constitutive equation which is constructed with the help of the different principles of formulation that are nowadays accepted, and which obeys the objectivity or rheological invariance requirement, an approximate heat conduction law which: (i) is essentially spatial (three independent components); (ii) exhibits the relaxation phenomenon necessary to suppress the paradox of infinite velocity of propagation; (iii) corresponds to a simple symmetry of the material (ie, *isotropy*, thus allowing the description of heat conduction in *all fluids* and in *isotropic solids*). We indicate three ways to approximate the general functional (on a time interval) constitutive equation which describes so-called 'simple' thermodeformable media (cf Maugin 1972b, c). Each way corresponds to a different continuity hypothesis concerning the functional. First, the heat conducting medium is supposed to possess a 'fading memory' (a well known concept in modern viscoelasticity theory, cf Truesdell and Noll 1965). By use of nonlinear functional analysis tools, this hypothesis leads, after approximation, to an integral constitutive equation for heat flux which, in turn, by inversion of the integral operator, yields a differential equation that exhibits the desired *relaxation* process. The second hypothesis concerns heat conducting media which are supposed to possess an 'infinitely short memory'. It yields a heat flux differential constitutive equation that exhibits a *retardation* phenomenon. Such a form should be excluded after the second principle of thermodynamics. The third hypothesis is that on which is based the description of so-called *rate-type materials* (cf Maugin 1973f) which are peculiar cases of 'simple' materials. It is shown that the expressions resulting from the first and second hypotheses are included in those resulting from the third. All constitutive equations and some of their generalizations indicated satisfy the principle of objectivity. It is also shown that the simplest heat flux constitutive equation which satisfies the requirements set forth such as causality, the second principle of thermodynamics, and the general principles of formulation, in particular, the principle of objectivity—equation (57)—, has a common limit—equation (61)—with Kranys' equation for *rigid heat conductors* in classical physics. It is further shown that, while the general constitutive equations on which the present study is based are likely to encompass the interactions between the different transport phenomena (eg, heat flow and viscosity in fluids), there is no coupling between these phenomena within the frame of the approximations considered here. We are thus led to doubt the validity of a heat flux equation recently proposed by Stewart (see equation (81) herein) for, so far, no coupling of the type indicated by him seems to be possible.

We see that the notion of thermal disturbances propagating at finite speed is linked to the concept of *functional* constitutive equation for the heat flux. We mention that different cases based on the last concept have recently been treated in nonrelativistic physics, in particular, by Gurtin and Pipkin (1968) in their nonlinear theory of rigid heat conductors with memory, and by McCarthy (1970a, b) in his theory of nonlinear thermomechanical materials with memory. However, these authors do not give approximations of the type given hereafter.

Below we recall the notation and some definitions useful in the sequel. In particular, we recall what we mean by a *locally released reference state* for the description of the deformation field of a general relativistic continuum. In § 2, the relativistic covariant gradient of temperature and the invariant gradient of temperature are introduced. The local form of the second principle of thermodynamics referred to as the *Clausius–Duhem inequality* in deformable media and the corresponding *dissipation inequality* are set forth in § 3. The notion of 'simple' thermodeformable medium and the principles of formulation of constitutive equations are given in § 4. The first approximation in the form of an 'integral' heat conduction law is given in § 5 after a precise mathematical definition of a material which has a 'fading memory'. The § 6 is devoted to the comparison of the equation so obtained with that of Kranys. The alternative formulation based on the notion of materials having an 'infinitely short memory' is given in § 7. The general case of rate-type thermodeformable materials and approximations corresponding to isotropic materials described by constitutive equations linear in the different *objective* variables are studied in § 8.

Notation (see the previous papers by the author). Let V^4 be the riemannian four-dimensional manifold of general relativity theory. Its metric $g_{\alpha\beta}(x^\lambda)$ is normal hyperbolic with lorentzian signature $+2$. x^λ, $\lambda = 1, 2, 3, 4$, is a local chart of $M = (V^4, g_{\alpha\beta})$ with x^4 time-like. In the sequel, Greek indices run from one to four and Latin indices run from one to three. ∇_α denotes the covariant derivative based on $g_{\alpha\beta}$. Commas denote partial differentiation. u^α is the four-velocity such that $g_{\alpha\beta}u^\alpha u^\beta = -c^2$. $\delta/\delta s$ indicates the *invariant* derivative in the u^α direction, ie, $(\delta/\delta s) = u^\alpha\nabla_\alpha$, with $u^\alpha = \delta x^\alpha/\delta s$. $\dot{u}^\alpha \equiv \delta u^\alpha/\delta s$ is the four-acceleration such that $g_{\alpha\beta}u^\alpha\dot{u}^\beta = 0$. $£_u$ indicates Lie differentiation with respect to the four-vector field u^α. Parentheses around a set of indices denote symmetrization†. Let $x^\alpha = \mathscr{X}^\alpha(X^K, s)$ be the diffeomorphism: $\mathbb{E}^3_R \times \mathbb{R} \mapsto M$ which describes the time-like trajectory $(\mathscr{C}_X K)$ of a 'particle' labelled (X^K) parametrized with respect to the proper time s of (X^K), a time-like parameter that increases monotonically along $(\mathscr{C}_X K)$. At any event point $\boldsymbol{P}(s) \in (\mathscr{C}_X K)$ $M_\perp(\boldsymbol{P})$ is the three-dimensional hypersurface locally orthogonal to $(\mathscr{C}_X K)$. Thus $M_\perp(\boldsymbol{P})$ is locally space-like. At a certain event point $\boldsymbol{P}_0(s = \tau_0) \in (\mathscr{C}_X K)$, let $\mathbb{E}^3_R$ be the three-dimensional euclidean hyperplane tangent to $M_\perp(\boldsymbol{P}_0)$. X^K are chosen, for the sake of simplicity, as cartesian coordinates in $\mathbb{E}^3_R$. They are constants along $(\mathscr{C}_X K)$. We say that $(\mathbb{E}^3_R, X^K)$ defines a *locally released reference state* (LRRS), for the orthonormal basis $\boldsymbol{G}_K$, $K = 1, 2, 3$, at $\boldsymbol{P}_0$ in $\mathbb{E}^3_R$ defines an *inertial frame* (for the 'particle' (X^K)) in which gravitation is removed (cf Maugin 1973e, f). This is an ideal state which, in agreement with the principle of equivalence, can be defined locally and not for an extended material body as a whole. At each event point $\boldsymbol{P}(s) \in (\mathscr{C}_X K)$, the operator

$$P_{\alpha\beta}(\boldsymbol{P}) \stackrel{\text{def}}{=} g_{\alpha\beta}(\boldsymbol{P}) + c^{-2}u_\alpha(\boldsymbol{P})u_\beta(\boldsymbol{P}) \tag{1}$$

which satisfies

$$P_{\alpha\beta}P^{\beta}_{\cdot\gamma} = P_{\alpha\gamma}, \qquad P_{\alpha\beta}u^\alpha = 0, \qquad P^{\cdot\alpha}_{\alpha} = 3, \tag{2}$$

is the operator of projection on to $M_\perp(\boldsymbol{P})$. A covariant or contravariant tensor defined on $M_\perp(\boldsymbol{P})$ is said to be 'orthogonal to u^α' and referred to, for short, as a PU *tensor field*. Such a tensor takes essentially spatial values. The operation of projection on to $M_\perp(\boldsymbol{P})$ obtained by application of the operator $P_{\alpha\beta}$ to all indices of a tensor field $A^{\alpha\beta\cdot\cdot}_{\cdots\cdot}$ defined at $\boldsymbol{P}(s)$ is denoted by $(A^{\alpha\beta\cdot\cdot}_{\cdots\cdot})_\perp$. For a PU tensor field $A^{\alpha\beta\cdot\cdot}_{\cdots\cdot}$, we have

$$(A^{\alpha\beta\cdot\cdot}_{\cdots\cdot})_\perp \equiv A^{\alpha\beta\cdot\cdot}_{\cdots\cdot}$$

since $P_{\alpha\beta}$ is idempotent and PU. Note that we shall use implicitly the following rule in the paper: if *any* tensor $B^{\cdot\cdot\cdot\cdot}_{\alpha\beta\cdot\cdot}$ is contracted with a PU tensor $A^{\alpha\beta\cdot\cdot}_{\cdots\cdot}$, then

$$A^{\alpha\beta\cdot\cdot}_{\cdots\cdot}B^{\cdot\cdot\cdot\cdot}_{\alpha\beta\cdot\cdot} = A^{\alpha\beta\cdot\cdot}_{\cdots\cdot}(B^{\cdot\cdot\cdot\cdot}_{\alpha\beta\cdot\cdot})_\perp.$$

Deformation (cf Maugin 1971a, c, 1973d, e). The direct and inverse relativistic deformation gradients with respect to the LRRS are defined, at $\boldsymbol{P}(s) \in (\mathscr{C}_X K)$, by

$$x^\alpha_K \equiv \left(\frac{\partial \mathscr{X}^\alpha}{\partial X^K}\right)_\perp \qquad x^\alpha_K u_\alpha = 0, \tag{3}$$

$$X^K_{\cdot,\alpha} \equiv \frac{\partial X^K}{\partial x^\alpha} \equiv \left(\frac{\partial X^K}{\partial x^\alpha}\right)_\perp \qquad X^K_{\cdot,\alpha}u^\alpha = 0. \tag{4}$$

† For example, $e_{(\alpha\beta)} = \frac{1}{2}(e_{\alpha\beta} + e_{\beta\alpha})$.

These are so-called bitensor fields defined on $\mathbb{E}_R^3 \otimes M$, and PU four-vector fields on V^4. The second of equations (4) holds for X^K and s are independent variables, ie, $(\delta/\delta s)X^K = 0$. We have

$$x_K^\alpha X^K_{.,\beta} = P^\alpha_{.\beta}, \qquad X^K_{.,\alpha} x_L^\alpha = \delta_L^K, \tag{5}$$

where δ_L^K is the Kronecker symbol in $\mathbb{E}_R^3$. Then the relativistic Green deformation tensor C_{KL} and its reciprocal $\overset{-1}{C}{}^{MN}$ are given by

$$C_{KL}(s) \equiv P_{\alpha\beta}(s) x_K^\alpha(s) x_L^\beta(s), \tag{6}$$

$$\overset{-1}{C}{}^{MN} C_{NL} = \delta_L^M, \qquad \overset{-1}{C}{}^{MN}(s) \equiv P^{\alpha\beta}(s) X^M_{.,\alpha}(s) X^N_{.,\beta}(s). \tag{7}$$

These are *invariants* in V^4 (although they depend on s) but cartesian tensors† in $\mathbb{E}_R^3$.

The relativistic velocity gradient tensor and the relativistic rate of strain tensor are PU tensor fields defined as

$$e_{\alpha\beta} \equiv P_{\alpha\gamma} P^\mu_{.\beta} \nabla_\mu u^\gamma = P^\mu_{.\beta} \nabla_\mu u_\alpha = (\nabla_\beta u_\alpha)_\perp, \tag{8}$$

and

$$\sigma_{\alpha\beta} \equiv e_{(\alpha\beta)} = (\nabla_{(\beta} u_{\alpha)})_\perp \tag{9}$$

respectively. One can verify that $\sigma_{\alpha\beta}$ is also expressible as

$$\sigma_{\alpha\beta} = \tfrac{1}{2} \pounds_u P_{\alpha\beta}, \tag{10}$$

and

$$\sigma_{\alpha\beta} = \frac{1}{2} \frac{\delta C_{KL}}{\delta s} X^K_{.,\alpha} X^L_{.,\beta}. \tag{11}$$

Reciprocally,

$$\frac{\delta C_{KL}}{\delta s} = 2\sigma_{\alpha\beta} x_K^\alpha x_L^\beta = (\pounds_u P_{\alpha\beta})_\perp x_K^\alpha x_L^\beta. \tag{12}$$

In fact, if $x^\alpha = \mathscr{X}^\alpha(X^K, s)$ is viewed as a coordinate transformation, then equation (6) defines the *transported by convection* of $P_{\alpha\beta}$ and $(\pounds_u ..)_\perp$ is nothing but the *convected derivative*—noted D_C—with respect to the proper time s. Indeed, let $A^{\alpha..\mu}_{.\beta}$ be a PU tensor field, then (cf Schouten 1954, p 106)

$$A^{K..Q}_{.L} \equiv A^{\alpha..\mu}_{.\beta} X^K_{.,\alpha} x_L^\beta \dots X^Q_{.,\mu},$$

$$\frac{\delta A^{K..Q}_{.L}}{\delta s} = (D_C A^{\alpha..\mu}_{.\beta}) X^K_{.,\alpha} x_L^\beta \dots X^Q_{.,\mu}, \tag{13}$$

$$D_C A^{\alpha..\mu}_{.\beta} \equiv (D_C A^{\alpha..\mu}_{.\beta})_\perp = \frac{\delta A^{K..Q}_{.L}}{\delta s} x_K^\alpha X^L_{.,\beta} \dots x_Q^\mu,$$

with

$$D_C A^{\alpha..\mu}_{.\beta} = (\pounds_u A^{\alpha..\mu}_{.\beta})_\perp$$
$$= \left(\frac{\delta A^{\alpha..\mu}_{.\beta}}{\delta s}\right)_\perp - A^{\nu..\mu}_{.\beta} e^\alpha_{.\nu} + A^{\alpha..\mu}_{.\gamma} e^\gamma_{.\beta} + \dots - A^{\alpha..\rho}_{.\beta} e^\mu_{.\rho}, \tag{14}$$

† Nevertheless, we keep the notation with subscript and superscript indices for these tensors. One may define other types of invariant and covariant deformation tensors; see Maugin 1973d, e in which a long bibliography on recent works on relativistic continuum mechanics is given.

since it is easily checked that (Maugin 1973b, f)

$$\left(\frac{\delta x^\alpha_K}{\delta s}\right)_\perp = e^\alpha_{.\lambda} x^\lambda_K, \qquad \left(\frac{\delta X^K_{.,\alpha}}{\delta s}\right)_\perp = -e^\beta_{.\alpha} X^K_{.,\beta}. \tag{15}$$

Finally, it is important to remark that every field can be expressed at any event point on $(\mathscr{C}_X K)$ as a function of X^K and s if the motion $x^\alpha = \mathscr{X}^\alpha(X^K, s)$ is supposed to be known up to that point. Also, if $A_{\alpha\beta..\mu}$ is a completely PU *covariant* tensor, then

$$(\pounds_u A_{\alpha\beta..\mu})_\perp \equiv \pounds_u A_{\alpha\beta..\mu}.$$

2. Relativistic gradient of temperature

We have shown elsewhere (Maugin 1973e) that, if one assumes, following Tolman and Ehrenfest (1930) (also Tolman 1934), that, in a *stationary* gravitational field, the thermodynamical equilibrium at event point $\boldsymbol{P}(s) \in (\mathscr{C}_X K)$ was represented by $T(x^\lambda) =$ constant where T is the '*invariant*' *temperature* (cf Landau and Lifshitz 1958), then the thermodynamical disequilibrium in the neighbourhood of $\boldsymbol{P}(s)$ was represented by the deviation from the formula $T_{,\alpha}|_{(\mathscr{C}_X K)} = 0$. In fact, with $T(x^\lambda)$ defined as (the metric is here written in so-called *adapted* coordinates)

$$T(x^\lambda) \equiv [-g_{44}(x^\lambda)]^{1/2}\theta(x^\lambda), \tag{16}$$

$$\theta > 0, \qquad \inf \theta = 0, \tag{17}$$

θ being the *proper* thermodynamical temperature, we have shown, by computing the expansion of T about $\boldsymbol{P}(s)$ along a spatial geodesic contained in $M_\perp(\boldsymbol{P})$ with the aid of Fermi coordinates, that

$$\delta T(x^\lambda) \simeq \overset{*}{\theta}_\alpha (\delta x^\alpha)_\perp = \theta_K \, \delta X^K, \tag{18}$$

in which

$$\overset{*}{\theta}_\alpha \equiv (\theta_{,\alpha} + c^{-2}\theta \dot{u}_\alpha)_\perp, \qquad \overset{*}{\theta}_\alpha u^\alpha = 0, \tag{19}$$

$$\theta_K \equiv x^\alpha_K \overset{*}{\theta}_\alpha, \qquad \overset{*}{\theta}_\alpha = X^K_{.,\alpha}\theta_K. \tag{20}$$

Equation (19) defines the *relativistic gradient of temperature* in covariant form while equation (20) defines the invariant (obtained by convection) gradient of temperature. The equations (18) assert that, up to terms of the second order, the thermodynamical disequilibrium in the vicinity of $\boldsymbol{P}(s) \in (\mathscr{C}_X K)$ is measured either by $\overset{*}{\theta}_\alpha$ or by θ_K. It is remarkable that the quantity $\overset{*}{\theta}_\alpha$ appears quite naturally in other approaches to relativistic dissipative materials, for instance, in those concerned with an approximate solution to the relativistic Boltzmann equation (cf Marle 1969, Vignon 1969; also Ehlers 1971, Stewart 1971). Also, the variable $\overset{*}{\theta}_\alpha$ appears naturally in the local statement of the second principle of thermodynamics for relativistic continua (cf Eringen 1970, Maugin 1971b).

3. Elements of thermodynamics of relativistic continua

Here we consider a pure phenomenological approach† and use no arguments of kinetic

† Other works on relativistic phenomenological thermodynamics not referred to in the text are, among many, those of Schöpf (1963), Bressan (1964), Müller (1969), Alts and Müller (1972).

theory (see Stewart 1971 for a kinetic-theory approach). We assume that there are neither electromagnetic fields nor spin so that the energy–momenta of the continuous medium can be described by a symmetric energy–momentum tensor $T^{\alpha\beta}$ which, at event point $\boldsymbol{P}(s) \in (\mathscr{C}_X K)$, admits a decomposition on to $M_\perp(\boldsymbol{P})$ and along u^α given by (cf Eckart 1940)†

$$T^{\alpha\beta} = \rho\left(1+\frac{\epsilon}{c^2}\right)u^\alpha u^\beta + \frac{2}{c^2}\, q^{(\alpha}u^{\beta)} - t^{\beta\alpha}, \qquad T^{[\alpha\beta]} = 0, \tag{21}$$

with

$$\begin{aligned} q^\alpha &\equiv -(T^{\alpha\beta}u_\beta)_\perp, & q^\alpha u_\alpha &= 0, & \\ t^{\beta\alpha} &\equiv -(T^{\alpha\beta})_\perp, & t^{\beta\alpha} &= t^{\alpha\beta}, & t^{\alpha\beta}u_\alpha = 0. \end{aligned} \tag{22}$$

Here ρ is the invariant relativistic density of matter, ϵ is the specific internal energy, q^α is the PU heat flux four-vector, and $t^{\beta\alpha}$ is the PU relativistic stress tensor. The conservation equations of relativistic continuum mechanics are

$$(\nabla_\beta T^{\alpha\beta})_\perp = 0, \qquad u_\alpha \nabla_\beta T^{\alpha\beta} = 0, \qquad \nabla_\alpha(\rho u^\alpha) = 0 \tag{23}$$

which are the first Cauchy equation of the motion, the energy conservation equation, and the relativistic continuity equation respectively. For dissipative processes, these are complemented by a local statement of the second principle of thermodynamics:

$$\nabla_\alpha \eta^\alpha \geqslant 0, \qquad \eta^\alpha \equiv \frac{q^\alpha}{\theta} + \eta u^\alpha, \tag{24}$$

where η^α is the entropy flux and η is the specific entropy. By using the second and the third of equations (23), and taking account of (21), it can be shown that (24) takes the 'Clausius–Duhem inequality' form (Maugin 1971b, Eringen 1970)‡

$$-\frac{1}{\theta}\left[\rho\left(\frac{\delta\psi}{\delta s}+\eta\frac{\delta\theta}{\delta s}\right) - t^{\beta\alpha}\sigma_{\alpha\beta} + \frac{1}{\theta}q^\alpha \overset{*}{\theta}_\alpha\right] \geqslant 0 \tag{25}$$

in which ψ is the specific Helmholtz free energy. It can be shown that the recoverable part ${}^{\mathrm{R}}t^{\beta\alpha}$ of the stress and η are derivable (see, for instance, Maugin 1973a) from the potential ψ so that the inequality (25) reduces to the 'production of entropy', or dissipation, form:

$$\theta p_{(\eta)} \equiv -\frac{1}{\theta}q^\alpha \overset{*}{\theta}_\alpha + {}^{\mathrm{D}}t^{\beta\alpha}\sigma_{\alpha\beta} \geqslant 0. \tag{26}$$

${}^{\mathrm{D}}t^{\beta\alpha}$ is the dissipative part of $t^{\beta\alpha}$. In agreement with the first of equations (13), we can introduce the invariant convected forms of q^α and ${}^{\mathrm{D}}t^{\beta\alpha}$:

$$Q^K \equiv X^K_{.,\alpha} q^\alpha, \qquad {}^{\mathrm{D}}\mathcal{T}^{KL} \equiv {}^{\mathrm{D}}t^{\beta\alpha} X^K_{.,\beta} X^L_{.,\alpha}. \tag{27}$$

Reciprocally,

$$q^\alpha = x^\alpha_K Q^K, \qquad {}^{\mathrm{D}}t^{\beta\alpha} = {}^{\mathrm{D}}\mathcal{T}^{KL} x^\beta_K x^\alpha_L. \tag{28}$$

† This expression is acceptable in the sense that the nonrelativistic limit of the second of equations (23) yields the usual equation of conservation of energy of classical continuum mechanics.

‡ The Clausius–Duhem inequality is given for more involved processes—electromagnetic media with spin—and used as a tool to deduce the recoverable parts of the constitutive equations in Maugin (1973a, c).

Then, on account of the expressions (20), (11) and (5), the inequality (26) can be written in a completely invariant form:

$$\theta p_{(\eta)} \equiv -\frac{1}{\theta}Q^K\theta_K + \tfrac{1}{2}{}^{\mathrm{D}}T^{KL}\frac{\delta C_{KL}}{\delta s} \geqslant 0. \tag{29}$$

We see from the inequalities (26) and (29) that $\overset{*}{\theta}_\alpha$ and $\sigma_{\alpha\beta}$ (respectively θ_K and $(\delta/\delta s)C_{KL}$) are the generalized affinities associated with the generalized fluxes q^α and ${}^{\mathrm{D}}t^{\beta\alpha}$ (respectively Q^K and ${}^{\mathrm{D}}T^{KL}$) respectively. In a naive theory of irreversible processes, one disregards the interactions between the different transport phenomena so that, in view of equation (26), one takes, for instance, q^α linear in $\overset{*}{\theta}_\alpha$. That is, for isotropic media,

$$q^\alpha = -\chi P^{\alpha\beta}\overset{*}{\theta}_\beta, \qquad \text{or} \qquad Q^K = -\chi\,\delta^{KL}\theta_L. \tag{30}$$

This is the covariant generalization of Fourier's linear law proposed by Eckart (1940)†. As ${}^{\mathrm{D}}t^{\beta\alpha}$ and q^α are supposed to be uncoupled, the heat conduction term verifies the second principle of thermodynamics in the forms (26) and (29) if and only if χ is greater or equal to zero since $\|\overset{*}{\boldsymbol{\theta}}\|^2_{M_\perp(\boldsymbol{P})} \equiv P^{\alpha\beta}(s)\overset{*}{\theta}_\alpha(s)\overset{*}{\theta}_\beta(s) \geqslant 0$ for all s ($\overset{*}{\theta}_\alpha$ is PU hence space-like), and $\|\boldsymbol{\theta}\|^2_{\mathbb{E}^3_R(P_0)} \equiv \delta^L_K\theta^K\theta_L \geqslant 0$. The equation (30) leads to the difficulties mentioned in the introduction. Eringen (1970, equation (6.30)), using a representation theorem for isotropic tensor-valued functions, has given an *exact nonlinear* expression for $q^\alpha = q^\alpha(\rho, \theta, \sigma_{\mu\lambda}, \overset{*}{\theta}_\beta)$ when the interactions between transport phenomena are not neglected in *isotropic* media. However, his expression would yield the same sort of difficulty and it is far too complex to be of any use. In the sequel, we shall focus our attention on *functional* (integral or differential) equations, and use the somewhat axiomatic approach proposed by the author (Maugin 1972a, b, c, 1973b, e, f, g) in relativistic continuum physics.

4. General functional form for the heat conduction law

By relativistic 'simple' materials‡, we understand materials for which the various constitutive *dependent* variables, such as the stresses, the heat flux, ..., depend functionally at most on the first gradients of the basic constitutive *independent* variables, the latter being, for instance, in the case of thermodeformable materials, and according to the old terminology, the 'causes', ie, the motion x^α and the thermodynamical temperature θ. The first gradients considered are carefully constructed quantities in a manner such that simultaneity and causality problems arising in curved space–time are solved. In fact, these are systematically obtained by expansion of the basic variables about their values at an event point $\boldsymbol{P}(s) \in (\mathscr{C}_X K)$ along a spatial geodesic (hence included in $M_\perp(\boldsymbol{P})$) issued from that point§. In the present case, these gradients are x^α_K and $\overset{*}{\theta}_\alpha$ defined by equations (3) and (19) respectively. According to the *principle of determinism*¶, the present 'effects' eg, the values of the dependent constitutive variables (stresses, heat flux) at event point $\boldsymbol{M}(s = \tau) \in (\mathscr{C}_X K)$ at which we examine the constitutive equations,

† It can also be obtained by using kinetic-theory arguments (cf Ehlers 1971, p 62).

‡ This notion was introduced in Maugin (1972a, b, c).

§ These gradients are constructed for different continuous media, eg, classical deformable media, oriented media, magnetized media, ..., in a paper by the author Maugin (1973e).

¶ The principles of formulation of constitutive equations in classical continuum physics are given in Truesdell and Noll (1965) and Eringen (1967, chap 5). The equivalent study for relativistic continuum physics will be given in a paper by the author (in preparation).

are determined by all *past* and *present* 'causes', ie, all values taken by the independent variables (x^{α}_K and $\overset{*}{\theta}_{\alpha}$) at event points $\boldsymbol{P}(s \leqslant \tau) \in (\mathscr{C}_X K)$ along the trajectory of (X^K) in space–time. Thus, even though a special configuration, the LRRS defined at $\boldsymbol{P}_0(s = \tau_0 < \tau) \in (\mathscr{C}_X K)$, is used to measure the deformation, the principle of determinism asserts that the constitutive equations of a *simple* thermodeformable material are, to start with, *general functionals* on $s \in (-\infty, \tau]$ of the arguments x^{α}_K and $\overset{*}{\theta}_{\alpha}$ (the constitutive equations cannot depend explicitly on the position x^{α} in space–time as would be easily shown in special relativity by considering invariance under space–time translations). That is,

$$t^{\beta\alpha}(\boldsymbol{M}(s = \tau)) = \mathscr{T}^{\beta\alpha}[x^{\gamma}_K(s), \overset{*}{\theta}_{\mu}(s) | \theta(\tau), X^K],$$

$$q^{\alpha}(\boldsymbol{M}(s = \tau)) = \mathscr{Q}^{\alpha}[x^{\gamma}_K(s), \overset{*}{\theta}_{\mu}(s) | \theta(\tau), X^K], \tag{31}$$

$s \in (-\infty, \tau]$. The same functional dependence holds true for ϵ and η. The notation $f(\tau) = \mathscr{F}[A(s)]$, $s \in (-\infty, \tau]$ means that the value of f at $\boldsymbol{M}(\tau)$ depends on all values taken by the argument A for $s \in (-\infty, \tau]$. Equivalently, the functional can also be written as $\tilde{\mathscr{F}}[A(s) | A(\tau)]$, $s \in (-\infty, \tau[$. In that case, it is considered to be a functional of A on $s \in (-\infty, \tau[$ and a function in the usual sense of $A(\tau)$. At this point, we need not specify the topological frame for the functionals (ie, the norm, the smoothness of the functionals, the existence of Fréchet derivatives ... ; these notions are needed to study approximations of functionals). In writing the formulae (31), we have made use of the *axiom of equipresence*. By this we mean that, unless it is forbidden by a general physical principle (such as the second principle of thermodynamics or the material symmetry of the medium), all 'causes' should contribute to all 'effects'. This axiom (cf Truesdell and Noll 1965) serves as a guideline in the formulation of constitutive equations for involved processes. Furthermore, the systematic introduction of seemingly too numerous independent variables in each constitutive equation allows, in fact, to take account of the different coupling effects which are overlooked in primitive treatments. For instance, the functional dependence introduced in the second of equations (31) means that the present value of the heat flux depends on all past values of the temperature gradient—this is quite natural—but also, on all past values of the deformation gradient, and this, in a *time-functional* way. This means that, if equations (31) describe transport phenomena, the different transport phenomena in general interact. For instance, viscosity can produce a heat flux, and a gradient of temperature can produce a stress and a deformation. Finally, the function dependence on the parameters X^K in equations (31) is introduced to take account of possible inhomogeneities. The degree of symmetry of the material is obviously not specified at this stage.

It is further required that constitutive equations for a relativistic thermodeformable material satisfy the so-called *principle of material frame indifference in relativity* (PMIR; also referred to as the principle of objectivity) of which two forms have been proposed by the author (Maugin 1972a, applied in Maugin 1972b, 1973b, and Maugin 1973f, g). The second form reads†: PMIR *constitutive equations*—represented by tensor fields with values on V^4—*of an ideal relativistic continuous deformable medium must be objective; that is, they must be invariant with respect to superposition of an arbitrary* local *Herglotz–Born rigid body motion.*

A *local* Herglotz–Born rigid body motion is such that $\sigma_{\alpha\beta}(s) \equiv 0$ for all s in an open neighbourhood $\mathscr{T}(\mathscr{C}_X K)$ of $(\mathscr{C}_X K)$. Oldroyd (1970) has proposed independently another

† We refer the reader to Maugin (1973f, g) for a precise mathematical statement of this principle. The statement of the PMIR given here appears to be a relativistic generalization of the classical principle of objectivity as given by, for instance, Truesdell and Noll (1965).

principle referred to as the *rheological invariance* requirement in which he advises the use of convected quantities in constitutive equations. However, in contrast with Oldroyd's statement, the present statement of the PMIR provides *necessary* and *sufficient* conditions to reduce the form of constitutive equations. The motivation for such a principle clearly appears in the results of its application. We have shown (Maugin 1973f) that, if the functional constitutive equations (31) are to satisfy the above statement of the PMIR, they must reduce, *necessarily* and *sufficiently*, to the forms:

$$t^{\beta\alpha}(M(s=\tau)) = x^{\beta}_{K}(\tau)\bar{T}^{KL}(\tau)x^{\alpha}_{L}(\tau), \qquad q^{\alpha}(M(s=\tau)) = x^{\alpha}_{K}(\tau)Q^{K}(\tau) \tag{32}$$

in which ($\xi \in (-\infty, \tau]$)

$$\bar{T}^{KL}(\tau) = \mathscr{T}^{KL}[C_{MN}(s), \theta_M(\xi)|C_{MN}(\tau), \theta(\tau), X^M],$$
$$Q^K(\tau) = \mathscr{Q}^K[C_{MN}(s), \theta_M(\xi)|C_{MN}(\tau), \theta(\tau), X^M], \tag{33}$$

$s \in (-\infty, \tau[$. C_{MN} and θ_M have been defined in equations (6) and (20) respectively. It follows from equations (33) that the rheological behaviour of the continuous material at event point $x^{\alpha}(X^K, \tau)$ only depends on information pertaining to the 'particle' (X^K). Hence, an observer co-moving with this 'particle' always determines the same type of response for the material submitted to the same type of solicitations; in particular, if he performs some experiment, he always determines the same values of the phenomenological constants needed to describe the behaviour of the material. We thus see the profound operational significance of the PMIR. Without such a principle, we are not sure at all to construct constitutive equations that can be experimentally studied in a laboratory. Of course, the notion of observer used above may appear as somewhat restrictive to those familiar with classical continuum physics. Nevertheless, it is obvious that, as a consequence of the curvature of space–time and general nonsimultaneity between two general event points, the statement of objectivity or material indifference can be but *local* in general relativity. Finally, it seems from the forms of equations (33) that the LRRS plays a peculiar role in the application of the PMIR. This is only because the LRRS was used to define the deformation field. It does not appear in the statement given above for the PMIR which is universal. In fact, we shall see below that we can free the formulation from a reference to the LRRS which, therefore, will have served as a useful intermediary. It is to be noted that the constitutive equations (33) obviously are *rheologically invariant* in the sense of Oldroyd (1970).

We now have to face two problems: (i) to find a manageable form instead of (33)—in particular for the second of these since we are interested in heat conduction—by specializing the functional Q^K; (ii) to specify the symmetry of the material. Below, we shall give three different ways to approximate the functionals. Regarding the second problem, we shall be content with the study of materials that possess the largest degree of symmetry, that is, *isotropy*. A remark is in order concerning this problem. The notion of material symmetry is definitely linked to our everyday notion of a three-dimensional euclidean physical world (eg, the notion of crystallographic group). It would be nonsense to speak of isotropy by using covariant formalism in curved space–time (although this is often done). It follows that the material symmetry of the continuous medium—ie, the study of the isomorphisms of a material 'particle' on to itself (Truesdell and Noll 1965)—even though it is relativistic, must be studied in a three-dimensional euclidean frame. The inertial frame provided by the LRRS is such a frame.

5. Integral form of a heat conduction law

We focus our attention on the second of equations (33) although the same method would also apply to the first of these. It proves more convenient to work with the quantity $\hat{Q}_L$ defined as

$$\hat{Q}_L(\tau) \equiv C_{LK}(\tau)Q^K(\tau), \qquad Q^K(\tau) = \overset{-1}{C}{}^{KL}(\tau)\hat{Q}_L(\tau). \tag{34}$$

On account of equations (27)–(28) and (6)–(7), it is a simple matter to show that

$$\hat{Q}_L(\tau) = x^\alpha_L(\tau)q_\alpha(\tau), \qquad q_\alpha(\tau) = X^K_{.,\alpha}(\tau)\hat{Q}_K(\tau). \tag{35}$$

Since Q^K is already a function of $C_{MN}(\tau)$ whose form is not specified, $\hat{Q}_L$ has a functional dependence of the same type as Q^K. That is,

$$\hat{Q}_L(\boldsymbol{M}(s = \tau)) = \mathfrak{C}_L[C_{MN}(s), \theta_M(\xi)|C_{MN}(\tau), \theta(\tau), X^K], \tag{36}$$
$$\xi \in (-\infty, \tau], s \in (-\infty, \tau[.$$

In order to simplify the algebra, we consider, in this section, that *there are no interactions between the different transport phenomena* so that we can neglect the dependence upon the history of the deformation tensor (temperature gradients are considered as the basic 'causes' of heat flux). This is equivalent to considering the medium as being *locally rigid*, ie, we can set $C_{KL}(s) = \delta_{KL}$ for all s. This is obviously wrong if the medium considered is, for instance, a fluid. We shall be satisfied with this simplifying hypothesis, which, as we shall see later, does not bring any damage to the derivation. In these conditions, the equation (36) reduces to

$$\hat{Q}_L(\tau) = \mathfrak{C}_L[\theta_K(\tau - s')|\theta(\tau)], \qquad s' \in [0, +\infty). \tag{37}$$

In this functional we have made a change of time variable by introducing $s' = \tau - \xi$. Moreover, we assumed, by dropping the explicit dependence on the parameters X^K, that the continuous medium was homogeneous.

We consider that $\hat{Q}_L$ depends functionally on θ_K only through the differential history $\bar{\theta}_K$ of θ_K defined as

$$\bar{\theta}_K(\tau, s') \equiv \theta_K(\tau) - \theta_K(\tau - s'), \qquad \bar{\theta}_K(\tau, 0) \equiv 0. \tag{38}$$

Hence (θ is a parameter),

$$\hat{Q}_L(\tau) = \mathscr{F}_L[\bar{\theta}_K(\tau, s')|\theta(\tau)], \qquad s' \in [0, +\infty). \tag{39}$$

In order to approximate the functional (39), we must specify the topological frame†. $\mathscr{H}^\oplus$ is a normed linear space of vector-valued functions $\boldsymbol{f}$ in $\mathbb{E}^3_R$ such that $\mathscr{H}^\oplus(\mathbb{E}^3_R) = \{\boldsymbol{f}(s'), s' \in [0, +\infty)\}$. $\mathscr{H}$ is the corresponding subspace such that $\mathscr{H}(\mathbb{E}^3_R) = \{\boldsymbol{f}(s'); s' \in]0, +\infty)\}$. Let $\boldsymbol{f}_r$ be the restriction of $\boldsymbol{f}$ in $]0, +\infty)$. We define the norm in $\mathscr{H}^\oplus$ as

$$\|\boldsymbol{f}\|^\oplus = \|\boldsymbol{f}_r\| + |\boldsymbol{f}(s' = 0)|. \tag{40}$$

A Hilbert space structure is given to $\mathscr{H}^\oplus$ by introducing an influence function $h(s')$ which is positive, monotonic, with domain $[0, +\infty)$, and decreasing fast enough to be

† The same method of approximation has been used in different papers, for instance, in Maugin (1971a, 1973h).

square integrable. We take $h(s'=0)=1$. We set

$$\|\boldsymbol{f}\|_h=\left(\int_0^\infty |\boldsymbol{f}(s')|^2 h(s')^2\,\mathrm{d}s'\right)^{1/2},$$
$$\|\boldsymbol{f}\|_h^\oplus=\|\boldsymbol{f}\|_h+|\boldsymbol{f}(s'=0)|. \tag{41}$$

In $\mathscr{H}_h$ and $\mathscr{H}_h^\oplus$, we have $\|\boldsymbol{f}\|_h<+\infty$ and $\|\boldsymbol{f}\|_h^\oplus<+\infty$ respectively. In $\mathscr{H}_h$, the inner product corresponding to the norm (41) is written

$$\langle f,g\rangle_h=\int_0^\infty \delta^{LK} f_L(s')g_K(s')h(s')^2\,\mathrm{d}s'. \tag{42}$$

A natural choice for h is (k is a parameter)

$$h(k,s')=\exp(-s'/2k),\qquad k>0. \tag{43}$$

This means that the distant past history of θ_K will not affect much the conduction behaviour of the material at event point $\boldsymbol{P}(s=\tau)\equiv\boldsymbol{P}(s'=0)$. Following the modern terminology (cf Truesdell and Noll 1965), we can say that the material has a fading 'memory'. We suppose that the functional $\mathscr{F}$ of equation (39) has for domain $[0,+\infty)$ an open set D of $\mathscr{H}_h^\oplus$ and possesses Fréchet derivatives $D^{\oplus n}\mathscr{F}[\bar\theta_K]$ up to the order N†. These are bounded, symmetric, n-linear forms in $\mathscr{H}_h^\oplus$, and they are continuous with respect to the product topology of their arguments. Considering the expansion of the functional $\mathscr{F}$ about the present history $\bar\theta_K(\tau,0)$ in $\mathscr{H}_h^\oplus$, we have

$$\hat{Q}_L(\tau)=f_L(\bar\theta_K(\tau,0))+\sum_{n=1}^{N}\frac{1}{n!}D^{\oplus n}\mathscr{F}_L[\bar\theta_K(\tau,s')|\theta(\tau)]+\mathscr{G}_L\{\mathscr{F}[\theta_K]\} \tag{44}$$

where the remainder $\mathscr{G}$ is a vector-valued functional of the order of $(\|\mathscr{F}\|_h)^N$. From the inequality (29), we see, by an argument of continuity, that Q^K vanishes whenever θ_K vanishes. It follows that the 'thermodynamical equilibrium' term $f_L(\bar\theta_K(\tau,0)=0)$ must vanish in equation (44) since its argument is identically zero. Further, the first Fréchet derivative $D^\oplus\mathscr{F}_L$ in the expansion (44), being a bounded—thus continuous—linear form in $\mathscr{H}_h^\oplus$, it can only be of the form (after the Fischer–Riesz theorem)

$$D^\oplus\mathscr{F}_L=\langle\Gamma_L^{\cdot K}(s'),\bar\theta_K(\tau,s')\rangle_h \tag{45}$$

which, after equation (38), reads

$$D^\oplus\mathscr{F}_L=\langle\Gamma_L^{\cdot K}(s'),\mathbf{1}\rangle_h\theta_K(\tau)-\langle\Gamma_L^{\cdot K}(s'),\theta_K(\tau-s')\rangle_h \tag{46}$$

where $\mathbf{1}$ is the unit linear operator. We consider *isotropic* media (in the LRRS) so that the linear operator $\Gamma_L^{\cdot K}$ must be spherical. That is,

$$\Gamma_L^{\cdot K}(s')=-\frac{\chi}{k}\delta_L^K \tag{47}$$

where χ is a constant. Thus Γ is independent of s' and $\pounds_u\Gamma_L^{\cdot K}=0$. χ may be dependent on θ but, in the present study, the latter is considered—cf equation (39)—as constant along $(\mathscr{C}_X K)$ although it varies from one trajectory to a neighbouring one. On account of equations (43) and (47), the expression (46) becomes

$$D^\oplus\mathscr{F}_L=-\chi(\theta)\left\{\theta_L(\tau)+\int_0^\infty\frac{\partial}{\partial s'}\left[\exp\left(-\frac{s'}{k}\right)\right]\theta_L(\tau-s')\,\mathrm{d}s'\right\}. \tag{48}$$

† Definitions of nonlinear functional analysis may be found in Rall (1971).

Defining

$$\bar{h}(k, s') \equiv h(k, s')^2 = \exp(-s'/k), \tag{49}$$

we can write this equation as

$$D^{\oplus}\mathscr{F}_L = -\chi(\theta)\left[\bar{h}(k, 0)\theta_L(\tau) + \int_0^{\infty}\left(\frac{\partial}{\partial s'}\bar{h}(k, s')\right)\theta_L(\tau - s')\,\mathrm{d}s'\right]. \tag{50}$$

The term within square brackets is none other than the convolution product on $\mathscr{D}'(\mathbb{R}^+)$, the index L being fixed, of $\mathscr{D}\bar{h}/\mathscr{D}\tau$ and θ_L in the sense of distribution theory (cf Schwartz 1966). Here $\mathscr{D}/\mathscr{D}\tau$ indicates differentiation with respect to the proper time in the sense of distribution theory. Given the properties of the product of convolution, we have

$$D^{\oplus}\mathscr{F}_L = -\chi(\theta)\frac{\mathscr{D}}{\mathscr{D}s}(\bar{h}(k, s) * \theta_L(s))\bigg|_{s=\tau}.$$

Hence we can write the equation (44) in the form

$$\hat{Q}_L(\tau) = -\chi(\theta)\mathscr{R}^{-1}(k)\{\theta_L\} + \sum_{n=2}^{N}\frac{1}{n!}D^{\oplus n}\mathscr{F}_L[\bar{\theta}_K|\theta] + \mathscr{G}_L\{\mathscr{F}[\bar{\theta}_K|\theta]\} \tag{51}$$

in which we used the notation

$$\mathscr{R}^{-1}(k)\{\theta_L\} = \frac{\mathscr{D}}{\mathscr{D}s}(\bar{h}(k, s) * \theta_L(s))\bigg|_{s=\tau}. \tag{52}$$

The operator $\mathscr{R}^{-1}$ which depends upon the parameter k is called the *inverse relaxation operator* for reasons explained below. If we neglect terms of the order of $(\|\mathscr{F}\|_h)$ in equation (51), we obtain the following *functional* heat conduction equation:

$$\hat{Q}_L(\tau) = -\chi(\theta)\mathscr{R}^{-1}(k)\{\theta_L\} \tag{53}$$

which, according to the expression (50), is of the *integral* type.

On account of equation (49), it is not difficult to see that the result (53) is none other than the integral—up to a constant that depends only on X^K and must be equal to zero—of the following differential equation (in the sense of distribution theory, cf Schwartz 1966), at any event point $\boldsymbol{P}(s) \in (\mathscr{C}_X K)$:

$$\mathscr{R}(k)\{\hat{Q}_L\} = -\chi(\theta)\theta_L(s). \tag{54}$$

$\mathscr{R}(k)$ is the *relaxation operator* defined as

$$\mathscr{R}(k) \equiv 1 + k\frac{\partial}{\partial s}. \tag{55}$$

k is the relaxation time. Here we have written $\partial/\partial s$ instead of $\delta/\delta s$ because $\hat{Q}_L$ is assumed to be expressed as a function of X^K and s. The equation (54) which may be considered as a *differential* constitutive equation for the convected heat flux is entirely expressed in terms of convected quantities. It is *objective* and is an approximation to the objective functional constitutive equation (39).

6. Comparison with Kranys' equation

It is not difficult to obtain the covariant differential equation that corresponds to the equation (54). Indeed, on account of equation (55), the first of equations (35), and the general formulae (13), we obtain from equation (54)

$$x^\alpha_L(q_\alpha + k \underset{u}{\pounds} q_\alpha) = -\chi(\theta)\theta_L. \tag{56}$$

Multiplying both sides of this equation by $X^L_{.,\beta}$ and using the first of (5) and the second of (20), we obtain the differential equation

$$R_\alpha^{.\beta}(k)q_\beta = -\chi(\theta)\overset{*}{\theta}_\alpha \tag{57}$$

with

$$R_\alpha^{.\beta}(k) \equiv P_\alpha^{.\beta}(1 + k\underset{u}{\pounds}). \tag{58}$$

We obtain the same equation if we apply the operator $R_{\alpha\beta}(k)$—the PU *relaxation operator*—to the contravariant four-vector q^β. The dissipation inequalities that correspond to the heat flux constitutive equations (53) and (57) are obtained by substituting the expressions of Q_L and $\overset{*}{\theta}_\alpha$ provided by these equations in the inequalities (26) and (29) respectively. We obtain

$$\theta p_{(n)} = \frac{\chi(\theta)}{\theta}\hat{\theta}^L \mathscr{R}^{-1}(k)\{\theta_L\} + \tfrac{1}{2}{}^{\mathrm{D}}T^{KL}\frac{\delta C_{KL}}{\delta s} \geqslant 0,$$

$$\theta p_{(n)} = [\theta\chi(\theta)]^{-1} q^\alpha(q_\alpha + k \underset{u}{\pounds} q_\alpha) + {}^{\mathrm{D}}t^{\beta\alpha}\sigma_{\alpha\beta} \geqslant 0. \tag{59}$$

In the first of these we have defined the invariant $\hat{\theta}^L$ by

$$\hat{\theta}^L \equiv X^L_{.,\alpha}\overset{*}{\theta}{}^\alpha = P^{\alpha\beta}X^L_{.,\alpha}\overset{*}{\theta}_\beta.$$

The inequalities (59) which are, respectively, an integral inequation and a differential inequation, must be verified at any event point $P(s) \in (\mathscr{C}_X K)$.

The equation (56) is *objective*—ie, it satisfies the PMIR—for all its terms are objective according to the statement of the PMIR. In particular, the projected Lie derivative of an objective PU tensor field—eg, $(\pounds_u q_\alpha)_\perp \equiv \pounds_u q_\alpha$, cf § 4—is an objective PU tensor field while the projected invariant derivative of the same tensor field—ie, $(\delta q_\alpha/\delta s)_\perp$—is not objective†. The heat conduction law *postulated* by Kranys (1966a, b, 1967) is not logically deduced as an approximation from a general constitutive equation as it is here in the case of equation (57). However its form is very close to that of equation (57), but it is not objective since it makes use of the non-objective quantity $\delta q_\alpha/\delta s$‡. Then it is not in agreement with the principles of formulation recalled in § 4. Nevertheless, if we neglect the nonzero invariant density of heat introduced by Kranys (also, Boillat 1971; see our comments in the introduction), and consider the purely spatial part of his equation, ie,

$$P_\alpha^{.\beta}\left(q_\beta + k\frac{\delta q_\beta}{\delta s}\right) = -\chi\overset{*}{\theta}_\alpha, \tag{60}$$

then, considering a local *adapted* chart $x^\alpha = (x^k, k = 1, 2, 3; x^4 = ct)$, such that $u^\alpha = (u^i = 0; u^4 = c)$, and neglecting products of gradients of the four-velocity with q_α, with $c \mapsto \infty$, we obtain from both equations (60) and (57) the following three-dimensional limit:

$$q_i + k\frac{\partial q_i}{\partial t} = -\chi\theta_{,i}, \qquad i = 1, 2, 3. \tag{61}$$

† These propositions are established in Maugin (1973f, appendix B).

‡ The same problem arises in the study of constitutive equations for elasticity in general relativity (compare Synge's and Bennoun–Carter–Quintana's constitutive equations, cf Maugin 1973f).

This is the equation originally proposed by Cattaneo (1948) and Vernotte (1958) in *rigid* heat conductors. Note that, in deformable bodies, the partial derivative $\partial/\partial t$ should be replaced by an objective time derivative if this constitutive equation has to be objective according to the principles of modern continuum physics (as it is the case with $\delta q_\alpha/\delta s$, the usual material time derivative would not be sufficient to satisfy this requirement; cf Truesdell and Noll 1965)†.

7. An alternative formulation

By considering a different assumption with respect to the functional constitutive equation (37), one can arrive at a simple form which differs from (57). Instead of the assumption represented by the equation (39), we consider that $\hat{Q}_L$ depends functionally on $\bar{\theta}_K$ and depends as a *function* in the usual sense, on θ_K, ie,

$$\hat{Q}_L(\tau) = \mathscr{F}_L[\bar{\theta}_K(\tau, s') | \theta_K(\tau), \theta(\tau)], \qquad s' \in [0, +\infty). \tag{62}$$

We make a smoothness hypothesis quite different from that considered in § 5. We approximate the functional (62) by assuming the following: the heat conduction material posseses an *infinitely* short 'memory' in the sense that only the very recent past history of θ_K affects the conduction behaviour of the material at event point $\boldsymbol{P}(s = \tau) \equiv \boldsymbol{P}(s' = 0)$. Mathematically, this reads

$$\hat{Q}_L(\tau) = \mathscr{F}_L[\bar{\theta}_K(\tau, s') | \theta_K(\tau), \theta(\tau)], \qquad s' \in [0, \varepsilon) \tag{63}$$

where ε is arbitrarily small. Then, in the past time neighbourhood $\mathscr{N}(\boldsymbol{P}) = [0, \varepsilon)$ of $\boldsymbol{P}(s' = 0) \in (\mathscr{C}_X K)$, the following Taylor series expansion at $s' = 0$ holds if θ_M is a sufficiently smooth function of s' $(s \equiv \tau - s')$:

$$\bar{\theta}_M(\tau, s') = \theta_M(\tau, 0) - \left(\theta_M(\tau, 0) + \sum_{p=1}^{P} \frac{(-1)^p}{p!} \frac{\partial^p \theta_M(s)}{\partial s^p}\bigg|_{s=\tau} (s')^p + \mathrm{O}((s')^{P+1}) \right)$$

In these conditions, the functional (63) can be approximated by the *function*

$$\hat{Q}_L(\tau) = g_L(\theta_M(\tau), \mathring{\theta}_M(\tau), \ddot{\theta}_M(\tau), \ldots; \theta(\tau)) \tag{64}$$

in which

$$\mathring{\theta}_M(\tau) = \frac{\partial}{\partial s} \theta_M(X^K, s)|_{s=\tau},$$

etc. In the *isotropic* case, a simple linear approximation to the function g_L is (time derivatives of an order greater than that of the first being neglected)

$$\hat{Q}_L(X^K, \tau) = -\chi(\theta) \mathscr{R}(k') \{\theta_L\} \tag{65}$$

with

$$\mathscr{R}(k') \equiv 1 + k' \frac{\partial}{\partial s}. \tag{66}$$

The operator $\mathscr{R}(k')$ which depends on the physical coefficient k' has a form similar to that of $\mathscr{R}(k)$ introduced in equation (55). However, whereas $\mathscr{R}(k)$ acts upon the 'effect'—the heat flux—in equation (54), $\mathscr{R}(k')$ acts upon the 'cause'—the thermodynamical

† The same problem arises in the study of classical hypo-elasticity which involves rates of the stress tensor (Truesdell and Noll 1965).

disequilibrium represented by the temperature gradient. Thus $\mathscr{R}(k')$ may be called a *retardation operator*. Performing on equation (65) the same transformation as that performed in the preceding section, we can obtain the corresponding covariant expression in the form

$$q_\alpha = -\chi(\theta)R_{\alpha}^{\cdot\beta}(k')\theta_\beta, \qquad R_{\alpha}^{\cdot\beta}(k') \equiv P_{\alpha}^{\cdot\beta}(1+k'£_u). \tag{67}$$

This constitutive equation is obviously objective. Bressan (1967, footnote p 209) *postulated* a similar form but his is not objective for he used the invariant derivative in lieu of the convected derivative. The differential equation corresponding to the dissipation inequality when q_α assumes the form (67) is easily obtained. We shall not give this expression.

One may ask if a constitutive equation of the form (67) solves the paradox of infinite propagation velocity of heat disturbances as equation (57) clearly does. In fact, the same question may be asked in regard with more general cases. For instance, an obvious generalization of both equations (57) and (67) would be

$$R_{\alpha}^{\cdot\beta}(k)q_\beta = -\chi R_{\alpha}^{\cdot\beta}(k')\overset{*}{\theta}_\beta, \tag{68}$$

of which a natural extension (J G Oldroyd, private communication) would seem to be

$$\left(\prod_{i=0}^{n} R_i(k_i)\right)q_\alpha = -\chi\left(\prod_{j=0}^{m} R_j(k'_j)\right)\overset{*}{\theta}_\alpha \tag{69}$$

in which

$$R_0(k_0 \equiv 0) \equiv 1,$$

$$R_i(k_i) \equiv 1+k_i£_u, \qquad k_i \neq 0, \qquad \text{for } i \geqslant 1,$$

and similar definitions hold for $R_j(k'_j)$. Equation (68) corresponds to the case $n = m = 1$, and equations (57) and (67) correspond to the cases $(n = 1, m = 0)$ and $(n = 0, m = 1)$ respectively. The question which may be raised is the following: while it seems quite natural to assume the existence of a relaxation process for heat conduction†, is it acceptable physically, in the case $(n = 0, m \geqslant 1)$, that there could be a nonzero heat flux produced by a finite rate of increase of temperature gradient even at zero temperature gradient, as equation (67) implies. After a classical argument (due to Coleman 1964) the inequality (26) implies that q_α vanishes whenever $\overset{*}{\theta}_\beta$ vanishes (at the same event point) if q_α is at least a C^1 function of $\overset{*}{\theta}_\beta$. Then it seems that the form (67) is prohibited. Moreover, from the mathematical viewpoint, it seems necessary that n be strictly greater than m in order to yield a strictly hyperbolic system of equations of evolution. This may be guessed by induction from the satisfying behaviour obtained with equation (57) but the general conjecture is presently outside the scope of this paper.

8. Rate-type constitutive equations

We now examine a third way to approximate the constitutive equations (33) for a 'simple' thermodeformable medium. In a former note (Maugin 1973b; also 1973f and 1973d, appendix), we have introduced a class of relativistic materials referred to as *rate-type materials*. They are defined as follows. First, we introduce the invariant $\hat{T}_{AB}$

† Maxwell (1867) had initially proposed a relaxation process for heat conduction. He later neglected the relaxation term by noting that the heat flux is rapidly established.

defined from $\overline{T}^{KL}$ by the equation

$$\hat{T}_{AB}(\tau) \equiv \overline{T}^{KL}(\tau)C_{AK}(\tau)C_{BL}(\tau) = t_{\alpha\beta}(\tau)x^{\alpha}_{A}(\tau)x^{\beta}_{B}(\tau). \tag{70}$$

Then, taking account of equations (70) and (34), we can define a 'simple' thermodeformable medium by the constitutive equations

$$\begin{aligned} \hat{T}_{AB}(\tau) &= \hat{\mathscr{T}}_{AB}[C_{MN}(s), \theta_M(\xi)|C_{MN}(\tau), \theta(\tau)], \\ \hat{Q}_L(\tau) &= \mathfrak{C}_L[C_{MN}(s), \theta_M(\xi)|C_{MN}(\tau), \theta(\tau)], \end{aligned} \tag{71}$$

with $s \in (-\infty, \tau[$ and $\xi \in (-\infty, \tau]$. These equations are equivalent to equations (33) for homogeneous media. After the principle of equipresence, similar functional constitutive equations hold for the specific internal energy ϵ and the specific entropy η. We recall that the continuous material considered is of the *rate-type* if and only if, for any dynamical process compatible with the functional equations (71), the *functions* $\hat{T}_{AB}(\tau)$, $\hat{Q}_L(\tau)$, $C_{MN}(\tau)$, $\theta_M(\tau)$ and $\theta(\tau)$ satisfy differential equations of the following form (that we write only for the typical dependent variable $\hat{Q}_L$; a similar, but not necessarily of the same order, differential equation holding for $\hat{T}_{AB}$. In general, we therefore have to deal with a system of differential equations)† at any event point $M(s = \tau) \in (\mathscr{C}_X K)$:

$$\left.\frac{\delta^{(m)}\hat{Q}_L(s)}{\delta s^{(m)}}\right|_{s=\tau} = \hat{g}_L\left(\left.\frac{\delta^{(r)}\hat{T}_{AB}(s)}{\delta s^{(r)}}\right|_{s=\tau}, \left.\frac{\delta^{(p)}C_{MN}(s)}{\delta s^{(p)}}\right|_{s=\tau}, \left.\frac{\delta^{(q)}\theta_M(s)}{\delta s^{(q)}}\right|_{s=\tau}, \left.\frac{\delta^{(n)}\hat{Q}_L(s)}{\delta s^{(n)}}\right|_{s=\tau}, \theta(\tau)\right), \tag{72}$$

$$(m) \geqslant 1, \qquad (n) = 0, 1, \ldots, m-1, \qquad p = 0, 1, \ldots, P,$$
$$q = 0, 1, \ldots, Q, \qquad r = 0, 1, \ldots, R,$$

in which $\hat{g}_L$ is a scalar-valued function in M (a vector-valued function in $\mathbb{E}^3_R$) and $\delta^{(p)}/\delta s^{(p)}$ indicates the pth invariant derivative. In equation (72), all dependent and independent variables expressed as functions of X^K and s are supposed to be sufficiently smooth in order to allow the existence of the different derivatives. $\hat{g}_L$ has such smoothness properties as to ensure that, for each prescribed sufficiently smooth functions $\theta_M(\tau)$, $\delta^{(p)}C_{MN}/\delta s^{(p)}$, $\delta^{(r)}\hat{T}_{AB}/\delta s^{(r)}, \ldots$, and prescribed initial data, for instance, $\hat{Q}_L(\tau_0), \ldots, \delta^{(m-1)}\hat{Q}_L/\delta s^{(m-1)}|_{s=\tau_0}$, the differential equation (72) has a unique solution $\hat{Q}_L(s = \tau)$. Note that, in general, it is not possible to reconstruct the corresponding functional equation (71, part two) from the relation (72). Obviously, the invariant derivatives of invariants are *objective*. Then the differential constitutive equation (72) which involves rates of the different dependent and reduced independent variables satisfies the PMIR. We can also write the equation (72) in the form

$$\hat{g}\left(\left.\frac{\delta^{(k)}\hat{Q}_L(s)}{\delta s^{(k)}}\right|_{s=\tau}, \left.\frac{\delta^{(r)}\hat{T}_{AB}(s)}{\delta s^{(r)}}\right|_{s=\tau}, \left.\frac{\delta^{(p)}C_{MN}(s)}{\delta s^{(p)}}\right|_{s=\tau}, \left.\frac{\delta^{(q)}\theta_M^{(s)}}{\delta s^{(q)}}\right|_{s=\tau}, \theta(\tau)\right) = 0 \tag{73}$$

where, now, $k = 0, 1, \ldots, m$. This is a general relation for all types of media. Interesting special cases are the following ones: (i) if the medium considered has a local Herglotz–Born rigid-body motion, then $C_{MN}(s) = \delta_{MN}$ and $\delta^{(p)}C_{MN}(s)/\delta s^{(p)} = 0, p > 0$, for all s along $(\mathscr{C}_X K)$ (these are direct consequences of the definition of a Herglotz–Born rigid-body

† The corresponding differential equation for $\hat{T}_{AB}$ leads, after approximations in which coupling with heat conduction is neglected, to the theory of relativistic hypo-elasticity (cf Maugin 1973b, f).

motion; see Maugin 1973e) and there is no need to consider stresses. Then the equation (73) reduces to

$$g\left(\frac{\delta^{(k)}\hat{Q}_L(s)}{\delta s^{(k)}}\bigg|_{s=\tau}, \frac{\delta^{(q)}\theta_M(s)}{\delta s^{(q)}}\bigg|_{s=\tau}, \theta(\tau)\right) = 0. \tag{74}$$

(ii) If the medium considered is a *fluid*, then it has no 'memory' whatsoever (of any peculiar configuration) and its deformation can be described only in terms of rates of deformation. However, its constitutive equations may depend on the value of the density (cf Truesdell and Noll 1965). Then, equation (73) reads

$$\mathfrak{g}\left(\frac{\delta^{(k)}\hat{Q}_L(s)}{\delta s^{(k)}}\bigg|_{s=\tau}, \frac{\delta^{(r)}\hat{T}_{AB}(s)}{\delta s^{(r)}}\bigg|_{s=\tau}, \frac{\delta^{(p)}C_{MN}(s)}{\delta s^{(p)}}\bigg|_{s=\tau}, \frac{\delta^{(q)}\theta_M(s)}{\delta s^{(q)}}\bigg|_{s=\tau}, \theta(\tau), \rho(\tau)\right) = 0 \tag{75}$$

where, now, $p = 1, \ldots, P$, ie, $C_{MN}(\tau)$ cannot appear as an argument. In fact, since there are no privileged configurations for the study of a fluid, we can take the present configuration at $M(s = \tau) \in (\mathscr{C}_X K)$. Noting the following general demonstrable results:

$$\begin{aligned}\frac{\delta^{(k)}A^{K..Q}_{.L}}{\delta s^{(k)}} &= (\underbrace{D_C \ldots D_C}_{k\text{ times}} A^{\alpha..\mu}_{.\beta})X^K_{.,\alpha}x^\beta_L \ldots X^Q_{.,\mu},\\ \underbrace{D_C \ldots D_C}_{k\text{ times}} A^{\alpha..\mu}_{.\beta} &\equiv (\underbrace{\pounds_u \ldots \pounds_u}_{k\text{ times}} A^{\alpha..\mu}_{.\beta})_\perp \equiv (\pounds^{(k)}_u A^{\alpha..\mu}_{.\beta})_\perp,\end{aligned} \tag{76}$$

in which $A^{\alpha..\mu}_{.\beta}$ and $A^{K..Q}_{.L}$ are the PU tensor field and the invariant considered in equation (13), we can write, upon using equations (12), (35), (70), (20), (13), and (76), the following equation that replaces the equation (75):

$$\mathfrak{g}((\pounds^{(k)}_u q_\alpha)_\perp, (\pounds^{(r)}_u t_{\alpha\beta})_\perp, (\pounds^{(p)}_u P_{\alpha\beta})_\perp, (\pounds^{(q)}_u \mathring{\theta}_\beta)_\perp, \theta(\tau), \rho(\tau)) = 0 \tag{77}$$

in which $k = 0, 1, \ldots, m$, $r = 0, 1, \ldots, R$, $p = 1, \ldots, P$, $q = 0, 1, \ldots, Q$. We have not introduced time derivatives of ρ since it is clear, after the equation of continuity (23, part three) which can be written, upon use of equations (9) and (10), as

$$\frac{\delta\rho}{\delta s} \equiv \pounds_u\rho = -\rho\sigma^\alpha_{.\alpha} = -\tfrac{1}{2}\rho P^{\alpha\beta}(\pounds_u P_{\alpha\beta})_\perp,$$

that all successive time derivatives of ρ can be represented by functions of ρ and the successive projected Lie derivatives of $P_{\alpha\beta}$ only. The latter are already taken into account in equation (77). It is now clear, in this formal approach, that, as a consequence of the use of the principle of equipresence, the rate-type heat flux constitutive equation (73)—or (77) in the cases of fluids (also linear hypo-elastic solids)—in general takes account of the interactions between different transport phenomena. In particular, we see from equation (77) that the rate of strain $\sigma_{\alpha\beta}$—in other words, the viscosity—might participate in the production of heat flux. However, if we consider a simple *linear* (in the different tensorial arguments) approximation to equation (77), not retaining product terms but considering that objective time derivatives of the same order are present for the different variables, we may write

$$q^\alpha + k^{\alpha\beta}\pounds_u q_\beta + \chi^{\alpha\beta}\mathring{\theta}_\beta + A^{\alpha\beta}\pounds_u\mathring{\theta}_\beta + B^{\alpha\beta\gamma}t_{\beta\gamma} + C^{\alpha\beta\gamma}\pounds_u t_{\beta\gamma} + D^{\alpha\beta\gamma}\sigma_{\beta\gamma} + E^{\alpha\beta\gamma}\pounds_u\sigma_{\beta\gamma} = 0 \tag{79}$$

where the tensorial coefficients introduced eventually depend on ρ and θ. If $t_{\beta\gamma}$ is the relativistic stress of a viscous newtonian fluid (ie, $t_{\beta\gamma}$ is linear in $\sigma_{\beta\gamma}$), then the terms involving $\sigma_{\beta\gamma}$ and $t_{\beta\gamma}$ on the one hand, and those involving $£_u\sigma_{\beta\gamma}$ and $£_u t_{\beta\gamma}$ on the other, in equation (79), can be gathered.

Now, for *isotropy* (which is necessarily the case for fluids, cf Truesdell and Noll 1965), we must have†

$$k^{\alpha\beta} = kP^{\alpha\beta}, \qquad \chi^{\alpha\beta} = \chi P^{\alpha\beta}, \qquad A^{\alpha\beta} = \chi k' P^{\alpha\beta} \tag{80}$$

with

$$£_u k = £_u \chi = £_u k' = 0,$$

and

$$B^{\alpha\beta\gamma} = C^{\alpha\beta\gamma} = D^{\alpha\beta\gamma} = E^{\alpha\beta\gamma} = 0,$$

for there exist no odd rank PU tensors such as $B^{\alpha\beta\gamma}, C^{\alpha\beta\gamma}, \ldots$, since these tensorial coefficients must be expressed by linear combinations of tensor products of the projection operator $P^{\alpha\beta}$. On account of the expressions (80), the equation (79) reduces to equation (68). As we have seen above, the second principle of thermodynamics further requires that k' be zero. It thus appears that we obtain the same approximation as that we would have obtained by considering the equation (74) for rigid heat conductors; in other words, the coupling between heat flux and, for instance, viscosity, cannot appear within the frame of the approximations considered.

If we had retained product terms in the approximation of equation (77), we could have objective terms of the form $q_\beta \sigma_\alpha^{\cdot\beta}, \sigma_\alpha^{\cdot\beta}\sigma_\beta^{\cdot\gamma} q_\gamma, \ldots$, but also, terms of the form $\sigma_\alpha^{\cdot\beta}\overset{*}{\theta}_\beta, \sigma_\alpha^{\cdot\beta}\sigma_\beta^{\cdot\gamma}\overset{*}{\theta}_\gamma$, etc, (cf Eringen 1970, equation (6.30)) in the resulting approximate equation. It is however to be noticed that the equations (71) being *functionals on a time interval*, there cannot be gradients—eg, $x_K^\gamma \nabla_\gamma \hat{T}_{AB}$—in the differential (with respect to time) equation (72). As a consequence, there cannot be any gradients of the form $\nabla_\gamma t_\alpha^{\cdot\beta}$—and more particularly $\nabla_\beta t_\alpha^{\cdot\beta}$—in the differential equation (77), explicitly‡. It follows that a heat flux constitutive equation of the type recently proposed by Stewart§, ie,

$$q_\alpha + k\frac{\delta q_\alpha}{\delta s} + l\nabla_\beta t_\alpha^{\cdot\beta} = -\chi\overset{*}{\theta}_\alpha \tag{81}$$

in which l is a new physical constant, does not enter in the frame of the theory presented here. Further, it is not objective since it makes use of the invariant derivative instead of the convected derivative. In fact, if we are confident in the value of the principles of formulation recalled in §4—these principles constitute a formalization of everyday experience, intuitive ideas, and circumstances in which simple theories that proved to be very efficient were established—then we may doubt the validity of equation (81) in spite of its apparently 'physical' derivation.

Acknowledgments

The author wishes to thank Professor J G Oldroyd of the University of Liverpool and an anonymous referee for helpful comments on a preliminary draft of this work.

† We have said before that the material symmetry should be studied in a euclidean three-dimensional frame. However, the results (80) are correct as it is easily verified.

‡ In fact, gradients can appear but only in the combinations $u^\gamma\nabla_\gamma t_{\alpha\beta}$ which are present in terms of the form $£_u t_{\alpha\beta}$ (compare with equation (14)).

§ Seminar at the International Colloqium on Gravitational Radiation held in Paris (June 1973).

References

Alts T and Müller I 1972 *Archs ration. Mech. Analysis* **48** 245–73
Boillat G 1971 *Relativistic Fluid Dynamics* ed C Cattaneo (Rome: Cremonese)
Bressan A 1964 *Rc. Semin. mat. Univ. Padova* **34** 1–73
—— 1967 *Nuovo Cim.* B **48** 201–21
Cattaneo C 1948 *Atti. Semin. mat. fis. Univ. Modena* **3** 3–21
—— 1958 *C.R. Acad. Sci., Paris* A **247** 431–3
Coleman B D 1964 *Archs ration. Mech. Analysis* **17** 1–46
Eckart C 1940 *Phys. Rev.* **58** 919–24
Ehlers J 1971 *General Relativity and Cosmology* ed K Sachs (New York: Academic Press)
Eringen A C 1967 *Mechanics of Continua* (New York: Wiley)
—— 1970 *A Critical Review of Thermodynamics* ed E B Stuart, B Gal-Or and J A Brainard (Baltimore: Mono Book)
Gurtin M E and Pipkin A C 1968 *Archs ration. Mech. Analysis* **31** 113–26
Kranys M 1966a *Nuovo Cim.* B **42** 51–70
—— 1966b *Phys. Lett* **22** 285–6
—— 1967 *Nuovo Cim.* B **50** 48–63
—— 1972a *Nuovo Cim.* B **88** 417
—— 1972b *Archs ration. Mech. Analysis* **48** 274–301
Landau L D and Lifshitz E M 1958 *Statistical Physics* (Oxford: Pergamon)
Mahjoub B 1971a *Ann. Inst. Henri Poincaré* **14** 133–47
—— 1971b *Ann. Inst. Henri Poincaré* **15** 153–68
Marle C 1969 *Ann. Inst. Henri Poincaré* **10** 127–94
Maugin G A 1971a *C.R. Acad. Sci., Paris* A **272** 1482–4
—— 1971b *C.R.Acad. Sci., Paris* A **273** 1010–2
—— 1971c *Ann. Inst. Henri Poincaré* A **15** 275–302
—— 1972a *C.R. Acad. Sci., Paris* A **275** 319–22
—— 1972b *C.R. Acad. Sci., Paris* A **275** 349–52
—— 1972c *C.R. Acad. Sci., Paris* A **275** 405–8
—— 1973a *J. Phys.* A: *Math., Nucl. Gen.* **6** 306–21
—— 1973b *C.R. Acad. Sci., Paris* A **276** 1027–30
—— 1973c *J. Phys.* A: *Math., Nucl. Gen.* **6** 1647–66
—— 1973d *Gen. Relativ. Gravitat. J.* **4** 241–72
—— 1973e *C. R. Acad. Sci., Paris* A **278** in the press
—— 1973f *Ann. Inst. Henri Poincaré* **20** in the press
—— 1973g *Proc. Int. Coll. C.N.R.S on Gravitational Radiation Paris* 1973 (Paris: C.N.R.S) to be published
—— 1973h *Int. J. Engng. Sci.* **11** (December)
Maxwell J C 1867 *Phil. Trans. R. Soc.* **157** 49–88
McCarthy M F 1970a *Int. J. Engng. Sci.* **8** 467–74
—— 1970b *Proc. Vibr. Probl.* **2** 123–33
Müller I 1969 *Archs ration. Mech. Analysis* **34** 259–82
Oldroyd J G 1970 *Proc. R. Soc.* A **316** 1–20
Rall L B (ed) 1971 *Nonlinear Functional Analysis and Applications* (New York: Academic Press)
Schöpf H G 1963 *Ann. Phys., Lpz.* **7** 377–95
Schouten J A 1954 *Ricci Calculus* (Berlin: Springer)
Schwartz L 1966 *Théorie des distributions* (Paris: Hermann)
Stewart J M 1971 *Non-equilibrium Relativistic Kinetic Theory* (Berlin: Springer)
Tolman R C 1934 *Relativistic Thermodynamics and Cosmology* (Oxford: Clarendon)
Tolman R C and Ehrenfest P 1930 *Phys. Rev.* **36** 1791
Truesdell C and Noll W 1965 *Handb. Phys.* vol III/3 ed S Flügge (Berlin: Springer)
Vernotte P 1958 *C.R. Acad. Sci., Paris* A **246** 3154–5
Vignon B 1969 *Ann. Inst. Henri Poincaré* **10** 31–66